The Possibility of Cosmic Acceleration via Spatial Averaging in Lemaître-Tolman-Bondi Models

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Abstract

We investigate the possible occurrence of a positive cosmic acceleration in a spatially averaged, expanding, unbound Lemaître-Tolman-Bondi cosmology. By studying an approximation in which the contribution of three-curvature dominates over the matter density, we construct numerical models which exhibit acceleration.

1 Introduction

We live in an inhomogeneous Universe, whose exact and complicated dynamics is described by Einstein's equations. It is generally assumed that when the spatial inhomogeneities are averaged over, the resulting Universe is described by the standard Friedmann equations for a homogeneous and isotropic cosmology. However, as is known [1], since Einstein equations are non-linear, the averaging over the inhomogeneous matter distribution will in general not yield the solution of Einstein equations which is described by the Friedmann-Robertson-Walker (FRW) metric. There will be corrections to the FRW solution, which could be small or large, and which could in principle lead to observational effects indicating a departure from standard FRW cosmology.

The possibility that the observed cosmic acceleration [2] is caused by the spatial averaging of the observed inhomogeneities, rather than by a dark energy, has been investigated and debated in the literature [3, 4, 5]. A systematic framework has been developed for describing the dynamics of a modified Friedmann universe, obtained after spatial averaging [6]. It has been suggested that, within the framework of standard cosmology with cold dark matter initial conditions, an explanation of the acceleration in terms of averaged inhomogeneities is unlikely to work [7]. However, it is perhaps fair to say that the matter cannot be treated as completely closed, and further studies are desirable [8].

The Lemaître-Tolman-Bondi (LTB) cosmology [9], being an exact solution of Einstein equations for inhomogeneous dust matter, provides a useful toy model for investigating the possible connection between acceleration and averaging of inhomogeneities. Various authors have examined different aspects of the model in this regard. The redshift-luminosity distance relation in an LTB model and its possible connection with cosmic acceleration, or the lack of it, have been studied by Celeriér [10], Alnes et al. [11] and by Vanderveld et al. [12], Sugiura et al. [13], Mustapha et al. [14], Iguchi et al. [15]. Nambu and Tanimoto [16] give examples of cosmic acceleration after averaging in an LTB model consisting of a contracting region

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and an expanding region. Other works which study cosmic acceleration in LTB models are those by Moffat [17], Mansouri [18], Chuang et al. [19], Räsänen [20] and Apostolopoulos et al. [21].

It has sometimes been suggested in the literature that both an expanding and a contracting region are needed for acceleration. In the present paper we will address a question which does not seem to have been addressed in the above-mentioned works: can spatial averaging in a universe consisting of a single expanding LTB region produce acceleration? We show that the answer is in the affirmative. We do this by considering a low density, curvature dominated unbound LTB model in which the contribution of matter density is negligible compared to the contribution of the curvature function. Further, we concentrate on the late time behaviour of such a model. As a result of this proposed simplification, the calculation of the acceleration of the averaged scale factor becomes relatively simpler and conclusions about acceleration can be drawn, for specific choices of the energy function.

In Section 2 of the paper we recall the effective FRW equations, resulting from spatial averaging in a dust dominated spacetime. In Section 3 we discuss spatial averaging for the marginally bound LTB model and point out there can be no acceleration in this case. The unbound LTB model is investigated in Section 4, in the approximation that the spatial curvature (equivalently, the energy function) dominates over the dust matter density, and numerical and analytical examples of acceleration are given.

2 Averaging in Dust Dominated Spacetime

For a general spacetime containing irrotational dust, the metric can be written in synchronous and comoving gauge[1],

$$ds^2 = -dt^2 + h_{ij}(\vec{x},t)dx^i dx^j \,. \tag{1}$$

The expansion tensor Θ^i_j is given by $\Theta^i_j \equiv (1/2)h^{ik}\dot{h}_{kj}$ where the dot refers to a derivative with respect to time t. The traceless symmetric shear tensor is defined as $\sigma^i_j \equiv \Theta^i_j - (\Theta/3)\delta^i_j$ where $\Theta = \Theta^i_i$ is the expansion scalar. The Einstein equations can be split [6] into a set of scalar equations and a set of vector and traceless tensor equations. The scalar equations are the Hamiltonian constraint (2a) and the evolution equation for Θ (2b),

$$^{(3)}\mathcal{R} + \frac{2}{3}\Theta^2 - 2\sigma^2 = 16\pi G\rho \tag{2a}$$

$$^{(3)}\mathcal{R} + \dot{\Theta} + \Theta^2 = 12\pi G\rho \tag{2b}$$

where the dot denotes derivative with respect to time t, $^{(3)}\mathcal{R}$ is the Ricci scalar of the 3-dimensional hypersurface of constant t and σ^2 is the rate of shear defined by $\sigma^2 \equiv (1/2)\sigma^i_j\sigma^j_i$. Eqns. (2a) and (2b) can be combined to give Raychaudhuri's equation

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2\sigma^2 + 4\pi G\rho = 0\,. \tag{3}$$

The continuity equation $\dot{\rho} = -\Theta\rho$ which gives the evolution of ρ, is consistent with Eqns. (2a), (2b). We only consider the scalar equations, since the spatial average of a scalar quantity can be defined in a gauge covariant manner within a given foliation of space-time. For the space-time described by (1), the spatial average of a scalar $\Psi(t,\vec{x})$ over a *comoving* domain $\mathcal{D}$ at time t is defined by

$$\langle\Psi\rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} d^3x\sqrt{h}\,\Psi \tag{4}$$

where h is the determinant of the 3-metric h_{ij} and $V_{\mathcal{D}}$ is the volume of the comoving domain given by $V_{\mathcal{D}} = \int_{\mathcal{D}} d^3x\sqrt{h}$.

[1]Latin indices take values 1..3, Greek indices take values 0..3. We set $c = 1$.

Spatial averaging is, by definition, not generally covariant. Thus the choice of foliation is relevant, and should be motivated on physical grounds. In the context of cosmology, averaging over freely-falling observers is a natural choice, especially when one intends to compare the results with standard FRW cosmology. Following the definition (4) the following commutation relation then holds [6]

$$\langle\Psi\rangle_{\mathcal{D}}^{\cdot} - \langle\dot{\Psi}\rangle_{\mathcal{D}} = \langle\Psi\Theta\rangle_{\mathcal{D}} - \langle\Psi\rangle_{\mathcal{D}}\langle\Theta\rangle_{\mathcal{D}} \tag{5}$$

which yields for the expansion scalar Θ

$$\langle\Theta\rangle_{\mathcal{D}}^{\cdot} - \langle\dot{\Theta}\rangle_{\mathcal{D}} = \langle\Theta^2\rangle_{\mathcal{D}} - \langle\Theta\rangle_{\mathcal{D}}^2 \,. \tag{6}$$

Introducing the dimensionless scale factor $a_{\mathcal{D}} \equiv \left(V_{\mathcal{D}}/V_{\mathcal{D}in}\right)^{1/3}$ normalized by the volume of the domain $\mathcal{D}$ at some initial time t_{in}, we can average the scalar Einstein equations (2a), (2b) and the continuity equation to obtain [6]

$$\langle\Theta\rangle_{\mathcal{D}} = 3\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \,, \tag{7a}$$

$$3\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 - 8\pi G\langle\rho\rangle_{\mathcal{D}} + \frac{1}{2}\langle\mathcal{R}\rangle_{\mathcal{D}} = -\frac{\mathcal{Q}_{\mathcal{D}}}{2} \,, \tag{7b}$$

$$3\left(\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right) + 4\pi G\langle\rho\rangle_{\mathcal{D}} = \mathcal{Q}_{\mathcal{D}} \,, \tag{7c}$$

$$\langle\rho\rangle_{\mathcal{D}}^{\cdot} = -\langle\Theta\rangle_{\mathcal{D}}\langle\rho\rangle_{\mathcal{D}} = -3\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\langle\rho\rangle_{\mathcal{D}} \,. \tag{7d}$$

Here $\langle\mathcal{R}\rangle_{\mathcal{D}}$, the average of the spatial Ricci scalar ${}^{(3)}\mathcal{R}$, is a domain dependent spatial constant. The 'backreaction' $\mathcal{Q}_{\mathcal{D}}$ is given by

$$\mathcal{Q}_{\mathcal{D}} \equiv \frac{2}{3}\left(\langle\Theta^2\rangle_{\mathcal{D}} - \langle\Theta\rangle_{\mathcal{D}}^2\right) - 2\langle\sigma^2\rangle_{\mathcal{D}} \tag{8}$$

and is also a spatial constant. The last equation (7d) simply reflects the fact that the mass contained in a comoving domain is constant by construction : the local continuity equation $\dot{\rho} = -\Theta\rho$ can be solved to give $\rho\sqrt{h} = \rho_0\sqrt{h_0}$ where the subscript 0 refers to some arbitrary reference time t_0. The mass $M_{\mathcal{D}}$ contained in a comoving domain $\mathcal{D}$ is then $M_{\mathcal{D}} = \int_{\mathcal{D}} \rho\sqrt{h}d^3x = \int_{\mathcal{D}} \rho_0\sqrt{h_0}d^3x =$ constant. Hence

$$\langle\rho\rangle_{\mathcal{D}} = \frac{M_{\mathcal{D}}}{V_{\mathcal{D}in}a_{\mathcal{D}}^3} \tag{9}$$

which is precisely what is implied by Eqn. (7d).

This averaging procedure can only be applied for spatial scalars, and hence only a subset of the Einstein equations can be smoothed out. As a result it may appear that the outcome of such an approach is severely restricted, and essentially incomplete due to the impossibility to analyse the full set of equations. However one should note that the cosmological parameters of interest are scalars, and the averaging of the exact scalar part of Einstein equations provides the requisite needed information. A more general strategy would be to consider the smoothing of tensors, which is beyond the scalar approach that certainly provides useful information, albeit not the full information.

Equations (7b), (7c) can be cast in a form which is immediately comparable with the standard FRW equations [22]. Namely,

$$\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = -\frac{4\pi G}{3}\left(\rho_{\rm eff} + 3P_{\rm eff}\right) \tag{10a}$$

$$\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 = \frac{8\pi G}{3}\rho_{\rm eff} \tag{10b}$$

with ρ_{eff} and P_{eff} defined as

$$\rho_{\text{eff}} = \langle\rho\rangle_{\mathcal{D}} - \frac{\mathcal{Q}_{\mathcal{D}}}{16\pi G} - \frac{\langle\mathcal{R}\rangle_{\mathcal{D}}}{16\pi G} \quad ; \quad P_{\text{eff}} = -\frac{\mathcal{Q}_{\mathcal{D}}}{16\pi G} + \frac{\langle\mathcal{R}\rangle_{\mathcal{D}}}{48\pi G} \,. \tag{11}$$

A necessary condition for (10a) to integrate to (10b) takes the form of the following differential equation involving $\mathcal{Q}_{\mathcal{D}}$ and $\langle\mathcal{R}\rangle_{\mathcal{D}}$

$$\dot{\mathcal{Q}}_{\mathcal{D}} + 6\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\mathcal{Q}_{\mathcal{D}} + \langle\mathcal{R}\rangle\dot{}_{\mathcal{D}} + 2\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\langle\mathcal{R}\rangle_{\mathcal{D}} = 0 \tag{12}$$

and the criterion to be met in order for the effective scale factor $a_{\mathcal{D}}$ to accelerate, is

$$\mathcal{Q}_{\mathcal{D}} > 4\pi G\langle\rho\rangle_{\mathcal{D}} \,. \tag{13}$$

3 The LTB Solution

The system of equations (10a), (10b) and (12) is only consistent, it does not close. For a completely general spacetime with dust, therefore, it is not possible to proceed with the analysis without making certain assumptions about the form of the functions $\mathcal{Q}_{\mathcal{D}}$ and $\langle\mathcal{R}\rangle_{\mathcal{D}}$ [6, 7]. For this reason, it becomes convenient to work with the LTB metric, an exact solution of the Einstein equations which is a toy model consisting of a spherically symmetric inhomogeneous dust dominated spacetime. In this section, we describe the LTB solution and apply to it the averaging procedure described above for the simplest, marginally bound case. In the next section we extend the analysis to the unbound LTB solution. The LTB metric for pressureless dust is given in the synchronous and comoving gauge, by

$$ds^2 = -dt^2 + \frac{R'^2(r,t)}{1+2E(r)}dr^2 + R^2(r,t)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \,. \tag{14}$$

The Einstein equations simplify to

$$\frac{1}{2}\dot{R}^2(r,t) - \frac{GM(r)}{R(r,t)} = E(r) \,, \tag{15a}$$

$$4\pi\rho(r,t) = \frac{M'(r)}{R'(r,t)R^2(r,t)} \,. \tag{15b}$$

Surfaces of constant r are 2−spheres having area $4\pi R^2(r,t)$. $\rho(r,t)$ is the energy density of dust, while $E(r)$ and $M(r)$ are arbitrary functions that arise on integrating the dynamical equations. Solutions can be found for three cases $E(r) > 0$, $E(r) = 0$ and $E(r) < 0$. We will restrict our attention to models in which $E(r)$ has the same sign, for all r. The solution for $E(r) = 0$ (the marginally bound case) has the particularly simple form

$$R(r,t) = \left(\frac{9GM(r)}{2}\right)^{1/3}(t - t_0(r))^{2/3} \,, \qquad \text{for } E(r) = 0 \,. \tag{16}$$

Here $t_0(r)$ is another arbitrary function arising from integration. The solution describes an expanding region, with the initial time t_{in} chosen such that $t > t_{in} \geq t_0(r)$ for all r. For the other two cases, the solutions can be written in parametric form

$$R = \frac{GM(r)}{2E(r)}(\cosh\eta - 1) \quad ; \quad t - t_0(r) = \frac{GM(r)}{(2E(r))^{3/2}}(\sinh\eta - \eta) \;,\; 0 \leq \eta < \infty \,, \quad \text{for } E(r) > 0 \,. \tag{17a}$$

$$R = \frac{GM(r)}{-2E(r)}(1-\cos\eta) \quad ; \quad t - t_0(r) = \frac{GM(r)}{(-2E(r))^{3/2}}(\eta - \sin\eta) \ , \ 0 \le \eta \le 2\pi, \quad \text{for } E(r) < 0 \, . \tag{17b}$$

In the unbound case ($E(r) > 0$), $R(r,t)$ increases monotonically with t, for every shell with label r. In the bound case ($E(r) < 0$), $R(r,t)$ increases to a maximum value $R_{max}(r)$ for each shell r and then decreases back to 0 in a finite time.

In all cases, there are two physically different free functions, although three arbitrary functions E, M and t_0 appear. One of the three represents the freedom to rescale the coordinate r. We use this freedom to set $R(r,t_{in}) = r$. To completely specify the solution, we specify the initial density $\rho_{in}(r)$ and the function $E(r)$. This specifies $M(r) = 4\pi\int_0^r \rho_{in}(\tilde{r})\tilde{r}^2 d\tilde{r}$ (which in the marginally bound case is interpreted as the mass contained in a comoving shell), and $t_0(r)$ can be solved for using Eqns. (16), (17a) or (17b) as the case may be, at time $t = t_{in}$.

3.1 Averaging the LTB Solution

The quantities defined in Sec. 2 can be computed for the LTB metric of Eqn. (14). The averages are computed over a spherical domain of radius $r_{\mathcal{D}}$, centered on the observer. Other choices of the averaging domain will possibly yield different results, however, the choice of a spherical domain seems natural for the spherically symmetric metric of Eqn. (14). For clarity, we suppress the r and t dependences of the various functions in the following

$$V_{\mathcal{D}} = 4\pi\int_0^{r_{\mathcal{D}}} \frac{R'R^2}{\sqrt{1+2E}}\,dr \ ; \ \langle\Theta\rangle_{\mathcal{D}} = \frac{4\pi}{V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} \frac{R^2\dot{R}' + 2R\dot{R}R'}{\sqrt{1+2E}}\,dr = \frac{\dot{V}_{\mathcal{D}}}{V_{\mathcal{D}}} \ ; \ M_{\mathcal{D}} = \int_0^{r_{\mathcal{D}}} \frac{M'}{\sqrt{1+2E}}\,dr \tag{18}$$

where a prime denotes a derivative with respect to r. Note that $M_{\mathcal{D}} \neq M(r_{\mathcal{D}})$ if $E \neq 0$. Only in the marginally bound case is the function $M(r)$ identified with the mass contained in the shell with label r. It is convenient to work with the combination $(2/3)\langle\Theta^2\rangle_{\mathcal{D}} - 2\langle\sigma^2\rangle_{\mathcal{D}}$ rather than evaluate the average rate of shear $\langle\sigma^2\rangle_{\mathcal{D}}$ separately. We define this to be $\mathcal{C}_{\mathcal{D}}$ and obtain

$$\mathcal{C}_{\mathcal{D}} \equiv \frac{2}{3}\langle\Theta^2\rangle_{\mathcal{D}} - 2\langle\sigma^2\rangle_{\mathcal{D}} = \frac{8\pi}{V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} \frac{2R\dot{R}\dot{R}' + \dot{R}^2R'}{\sqrt{1+2E}}\,dr \, . \tag{19}$$

3.2 The marginally bound case - vanishing backreaction

The results of the previous subsection hold for all classes of the LTB solution, provided the averaging domain is spherically symmetric about the center. Now consider the marginally bound case $E(r) = 0$ for all r. The algebra in this case becomes very simple, and the backreaction can be computed analytically. We will show next that for a single domain with $E(r) = 0$ throughout, the backreaction $\mathcal{Q}_{\mathcal{D}}$ is, in fact, zero. Also, the average spatial curvature $\langle\mathcal{R}\rangle_{\mathcal{D}}$ is zero (which is expected by inspection of the metric (14) if we note that a spatially uniform initial density profile in the LTB solution in this case yields the corresponding FRW solution). As described earlier we have

$$M(r) = 4\pi\int_0^r \rho_{in}(\tilde{r})\tilde{r}^2 d\tilde{r} \quad ; \quad t_0(r) = t_{in} - \frac{r^{3/2}}{3}\sqrt{\frac{2}{GM(r)}} \, . \tag{20}$$

To obtain the second equation we have used Eqn. (16) at time $t = t_{in}$ with the condition $R(r,t_{in}) = r$. Some algebra then yields the following results

$$\dot{R} = \frac{2}{3}\left(\frac{9GM(r)}{2}\right)^{1/3}(t - t_0(r))^{-1/3} \tag{21a}$$

$$R' = \frac{2}{3}\left(\frac{9GM(r)}{2}\right)^{1/3}(t-t_0(r))^{-1/3}\left[\frac{M'(r)}{2M(r)}(t-t_0(r)) - t_0'(r)\right] \tag{21b}$$

$$\dot{R}' = \frac{2}{9}\left(\frac{9GM(r)}{2}\right)^{1/3}(t-t_0(r))^{-4/3}\left[\frac{M'(r)}{M(r)}(t-t_0(r)) + t_0'(r)\right] \tag{21c}$$

$$V_{\mathcal{D}} = 6\pi G M_{\mathcal{D}}\left(t-t_0(r_{\mathcal{D}})\right)^2 \;\Rightarrow\; a_{\mathcal{D}}(t) = \left(\frac{t-t_0(r_{\mathcal{D}})}{t_{in}-t_0(r_{\mathcal{D}})}\right)^{2/3} \tag{21d}$$

$$\langle\Theta\rangle_{\mathcal{D}} = \frac{2}{t-t_0(r_{\mathcal{D}})} \quad ; \quad \mathcal{C}_{\mathcal{D}} = \frac{8}{3\left(t-t_0(r_{\mathcal{D}})\right)^2}\,. \tag{21e}$$

Since the solution was constructed assuming $t > t_0(r)$ for all r, Eqn. (21d) immediately shows that $\ddot{a}_{\mathcal{D}} < 0$ and hence acceleration is not possible in this case. Further, Eqn. (21e) shows that $Q_{\mathcal{D}} = \mathcal{C}_{\mathcal{D}} - (2/3)\langle\Theta\rangle_{\mathcal{D}}^2 = 0$. Thus the backreaction term vanishes for a region described by the marginally bound LTB solution. This result is not unexpected. We note that mathematically, the General Relativistic equations (15a) and (15b) describing the evolution of a spherical dust cloud are identical to the corresponding Newtonian equations. It has been shown by Buchert, et. al. [23] that the backreaction $\mathcal{Q}_{\mathcal{D}}$ in a spherically symmetric *Newtonian* model of dust, must vanish. Further, we note that *in the marginally bound case*, the mathematical expressions for the averaged quantities defined earlier coincide with their corresponding Newtonian analogues. Hence, for the fully relativistic (marginally bound) case also, the backreaction must vanish.

The spatial Ricci scalar and its spatial average for a general $E(r)$ are given by

$$^{(3)}\mathcal{R} = -\frac{4}{R^2}\left(E + \frac{E'R}{R'}\right) \quad ; \quad \langle\mathcal{R}\rangle_{\mathcal{D}} = -\frac{16\pi}{V_{\mathcal{D}}}\int_0^{r_{\mathcal{D}}} \frac{\frac{\partial}{\partial r}(ER)}{\sqrt{1+2E}}\,dr \tag{22}$$

which shows that $^{(3)}\mathcal{R}$ and hence $\langle\mathcal{R}\rangle_{\mathcal{D}}$ vanish in the marginally bound case. This is consistent with the requirement of Eqn. (12).

4 The unbound LTB solution

Since the solution with zero spatial curvature fails to produce a non-trivial backreaction, we consider next the opposite extreme - a curvature dominated solution in which the contribution to the Einstein equations due to matter is much smaller than that due to spatial curvature. Before describing the construction of such a solution, we present a general treatment of regularity conditions which an unbound LTB model must satisfy.

4.1 Regularity conditions on unbound LTB models

Consider the class of unbound LTB models given by (17a). The functions $M(r)$ and $E(r)$ are to be specified by initial conditions at $t = t_{in}$, and the choice of scaling $R(r, t_{in}) = r$ fixes $t_0(r)$ as

$$t_0(r) = t_{in} - \frac{GM(r)}{(2E(r))^{3/2}}\left(\sinh\eta_{in}(r) - \eta_{in}(r)\right) \quad ; \quad \cosh\eta_{in}(r) - 1 = \frac{2E(r)r}{GM(r)}\,. \tag{23}$$

The regularity conditions imposed on this model, and their consequences, are as follows

- *No evolution at the symmetry centre*:
  This is required in order to maintain spherical symmetry about the same point at all times, and translates as $\dot{R}(0,t) = 0$ for all t. The right hand side of Eqn. (15a) must therefore vanish in the limit $r \to 0$. Since the functions involved are non-negative, we assume that we can write

$$E(r \to 0) \sim r^{\delta} \ , \ \delta > 0 \quad ; \quad M(r \to 0) \sim r^{\alpha} \ ; \ R(r \to 0, t) \sim r^{\beta} f(t) \ , \ \alpha > \beta \geq 0 \,. \tag{24}$$

  Consistency requires β to be constant, and our scaling choice further requires $\beta = 1$. We do not require the exponents δ and α to necessarily be integers.

- *No shell-crossing singularities*:
  Physically, we demand that an outer shell (labelled by a larger value of r) have a larger area radius R than an inner shell, at any time t. Unphysical shell-crossing singularities arise when this condition is not met. Mathematically, this requires

$$R'(r,t) > 0 \quad \text{for all } r\text{, for all } t. \tag{25}$$

- *Regularity of energy density*:
  We demand that the energy density $\rho(r,t)$ remain finite and strictly positive for all values of r and t. Combining this with Eqns. (15b) and (25) gives (assuming that R' is finite for all r and since $\beta = 1$)

$$\lim_{r \to 0} \rho(r,t) = \text{ finite} \Rightarrow \ \alpha - 1 - 2\beta = 0 \Rightarrow \ \alpha = 3 \,. \tag{26}$$

- *No trapped shells*:
  In order for an expanding shell to not be trapped initially, it must satisfy the condition $r > 2GM(r)$. Near the regular center, this condition is automatically satisfied independent of the exact form of $M(r)$, since there $M \sim r^3$.

Consider now the function $t_0(r)$ given by (23). By observing the behaviour of the functions $(\cosh \eta_{in} - 1)$ and $(\sinh \eta_{in} - \eta_{in})$ for values of δ equal to, less than, and greater than 2, it is easy to check that $t_0(r)$ is finite at $r = 0$ for all values of δ. However, this involves the assumption that $M(r)$ is positive for $r \neq 0$. In the limit of $M \to 0$ for all r, we find

$$R \simeq \ \sqrt{2E} \, (t - t_0(r)) \quad ; \quad t_0(r) \simeq \ t_{in} - \frac{r}{\sqrt{2E}} \,. \tag{27}$$

Although now, in the limit $r \to 0$, $t_0(r)$ is finite only when $\delta \leq 2$, it will turn out that the integrals involved in the averaging procedure are insensitive to the behaviour of $t_0(r)$ in the $r \to 0$ limit, and remain well defined for all positive values of δ. The expression for ${}^{(3)}\mathcal{R}$ in (22) indicates that the spatial Ricci scalar diverges as $r \to 0$ unless $\delta \geq 2$. However, we note that the spatial Ricci scalar is not a fully covariant quantity and depends on our choice of time slicing. The *four*-dimensional Ricci scalar, obtained after taking the trace of the Einstein equations as ${}^{(4)}\mathcal{R} = 8\pi G\rho(r,t)$ is finite at the origin irrespective of the behaviour of $E(r)$. It is interesting to see how this cancellation occurs. We have

$${}^{(4)}\mathcal{R} = \ {}^{(3)}\mathcal{R} + 2\left\{\left(\frac{\dot{R}}{R}\right)^2 + 2\frac{\ddot{R}}{R}\right\} + 2\left\{\frac{\ddot{R}'}{R'} + 2\frac{\dot{R}\dot{R}'}{RR'}\right\} \quad ; \quad {}^{(3)}\mathcal{R} = -\frac{4}{R^2}\left\{E + \frac{E'R}{R'}\right\} \tag{28}$$

On using the Einstein equation (15a) we obtain

$$2\left\{\left(\frac{\dot{R}}{R}\right)^2 + 2\frac{\ddot{R}}{R}\right\} = 4\frac{E}{R^2} \quad ; \quad 2\left\{\frac{\ddot{R}'}{R'} + 2\frac{\dot{R}\dot{R}'}{RR'}\right\} = 2\frac{GM'}{R^2R'} + 4\frac{E'}{RR'} \tag{29}$$

which neatly cancels the contribution from ${}^{(3)}\mathcal{R}$, leaving precisely $8\pi G\rho(r,t)$ after applying the second Einstein equation (15b). Hence the 4−dimensional Ricci scalar does not impose any further restrictions on the form of $E(r)$. The fact that the origin is well behaved can also be seen from the behaviour of the Kretschmann scalar, given by [25]

$$ {}^{(4)}\mathcal{R}_{\mu\nu\sigma\rho}\,{}^{(4)}\mathcal{R}^{\mu\nu\sigma\rho} = 12\frac{G^2M'^2}{R^4R'^2} - 32\frac{G^2MM'}{R^5R'} + 48\frac{G^2M^2}{R^6} \,. \tag{30} $$

A condition on the value of δ *is* obtained, however, by the regularity of the energy density $\rho(r,t)$, which assumes that $R'(r,t)$ is not only positive, but also finite for all r and t. Equation (27) shows that unless $\delta = 2$, R' either diverges or vanishes at the center, violating this regularity condition.

4.2 Late time solution and curvature dominated unbound models

The function $R(r,t)$ is an increasing function of time in all the unbound models described by (17a). The Einstein equation (15a) then shows that for sufficiently late times $t \gg t_{in}$, neglecting the term involving $1/R$, *all* unbound models have the approximate solution given by (27). If on the other hand, we start with a model which satisfies $GM(r)/(rE(r)) \ll 1$ for all r, then since our scaling assumes that $R = r$ at $t = t_{in}$, we will have $GM(r)/(R(r,t)E(r)) \ll 1$ for all r and for all $t \geq t_{in}$, and (27) is then an approximate solution at all times, the approximation becoming better as t increases. To make this idea more precise, consider the closed form expression for t in terms of R obtained by integrating Eqn. (15a) [24]

$$ t - t_0(r) = \frac{R^{3/2}}{(2GM)^{1/2}}\,F\left(\frac{ER}{GM}\right) \quad ; \quad F(x) \equiv \frac{1}{x}(1+x)^{1/2} - \frac{1}{x^{3/2}}\sinh^{-1}\left(x^{1/2}\right) \,. \tag{31} $$

Hence, imposing $R(r,t_{in}) = r$ we have

$$ t_0(r) = t_{in} - \frac{r^{3/2}}{(2GM)^{1/2}}F\left(\frac{Er}{GM}\right) \,. \tag{32} $$

Let us write $GM(r)/E(r) \equiv \epsilon G\tilde{M}(r)/E(r) \equiv \epsilon g(r)$ where ϵ is a dimensionless positive number whose value we can control. This relation also defines the functions $\tilde{M}(r)$ and $g(r)$. We can rewrite (31) for $\epsilon \ll 1$ as

$$ \sqrt{2E}\,(t - t_0(r)) = R\left\{1 + \frac{\epsilon g}{2R}\ln\left(\frac{\epsilon g}{R}\right) - \frac{\epsilon g}{R}\left(\ln 2 - \frac{1}{2}\right) + \mathcal{O}\left(\left(\frac{\epsilon g}{R}\right)^2\right)\right\} \,. \tag{33} $$

Here $\mathcal{O}(x^2)$ represents a power series beginning with a term of order x^2, and we have used a binomial expansion in $\epsilon g(r)/R$ and the asymptotic expansion for the inverse hyperbolic sine given by (as $x \to 0$) [26]

$$ \sinh^{-1}\left(\frac{1}{x}\right) = \ln 2 - \ln x + \frac{1}{4}x^2 + \mathcal{O}\left(x^4\right) \,. \tag{34} $$

The terms in Eqn. (33) involving ϵ vanish as $\epsilon \to 0$, although the expression in (33) cannot be inverted to get $R = R(r,t)$, due to the presence of the logarithm. We can, however, make the following statement. Provided the function $g(r)/r$ is finite for all values of r, then given any starting time t_{in}, we can choose ϵ small enough that the terms involving ϵ on the right hand side of Eqn. (33) are all negligible compared to unity. Then, since R increases with time, these terms will always be negligible compared to unity. Alternatively, given some $\epsilon g/r = GM/(Er)$ which is finite for all r, one can always wait for a sufficiently long time, and find that the ϵ dependent terms become smaller compared to unity. In this case, we need not even assume that ϵ is small. It is in this sense that the approximation involved in writing the equations in (27) becomes better as t increases (with the caveat that if ϵ is not small, then $t_0(r)$ must be given by the full expression (32) and not the approximation of (27)). This shows that the first of equations (27) is the correct late time solution for all unbound models, with the second being a good approximation when ϵ is small. The condition that for some $\epsilon > 0$, $g(r)/r$ be finite for all r, and in particular as $r \to 0$, implies that $\delta \leq 2$ where δ is defined in (24). This is not inconsistent with the requirement $\delta = 2$ imposed by the criterion of regularity of energy density.

Consider now a model which begins with negligible matter ($\epsilon \to 0$) *and* in which we have waited for a sufficiently long time ($t \gg t_{in}$). Eliminating $t_0(r)$ from (27) the approximate solution becomes

$$R(r,t) = \sqrt{2E(r)}\,(t - \lambda_t t_{in}) + \lambda_r r \tag{35}$$

where we have introduced two placeholders λ_r and λ_t which will remind us of the relative magnitudes of various terms. We will ultimately set $\lambda_r = \lambda_t = 1$. Substituting for R in the expression for the domain volume $V_{\mathcal{D}}$ in Eqn. (18), we find

$$V_{\mathcal{D}} = (t - \lambda_t t_{in})^3 \mathcal{I}_E + \lambda_r (t - \lambda_t t_{in})^2 \mathcal{I}_{Er} + \lambda_r^2 (t - \lambda_t t_{in}) \mathcal{I}_{Er^2} + \lambda_r^3 \mathcal{I}_{r^2} \tag{36}$$

where we have defined the domain dependent integrals

$$\mathcal{I}_E = 4\pi \int_0^{r_{\mathcal{D}}} \frac{\sqrt{2E}E'}{\sqrt{1+2E}}\, dr \quad ; \quad \mathcal{I}_{Er} = 4\pi \int_0^{r_{\mathcal{D}}} \frac{(r\cdot 2E)'}{\sqrt{1+2E}}\, dr$$

$$\mathcal{I}_{Er^2} = 4\pi \int_0^{r_{\mathcal{D}}} \frac{\left(r^2\cdot\sqrt{2E}\right)'}{\sqrt{1+2E}}\, dr \quad ; \quad \mathcal{I}_{r^2} = 4\pi \int_0^{r_{\mathcal{D}}} \frac{r^2}{\sqrt{1+2E}}\, dr\,. \tag{37}$$

The sum of the exponents of λ_r and λ_t in each term in (36) indicates the relative order of that term with respect to the leading t^3 term. This approach of treating some terms as small compared to others is valid since the various integrals which multiply the powers of t, are all finite and non-zero.

We note that the solution in Eqn. (27) actually corresponds to Minkowski spacetime, since in this limit the matter content has been neglected. The corresponding Riemann tensor and Kretschmann scalar are exactly zero. The constant time three-spaces are hypersurfaces of negative curvature, with the three-curvature being determined by the function $E(r)$. The 'FRW' limit of this solution is in fact the Milne universe; the solution (27) could hence be thought of as the 'Tolman-Bondi' type generalization of the Milne universe. For our purpose, it is not a problem that the solution describes Minkowski spacetime - we know from the initial conditions that dust matter is present, only its density is negligible compared to the curvature term. The form of the solution then allows us to easily determine if the average scale factor $a_{\mathcal{D}}$ undergoes acceleration. We demonstrate this with explicit examples in the next subsection. Subsequently, we argue that if a small amount of matter is introduced, so as to introduce departure from Minkowski spacetime, the sign of the acceleration of $a_{\mathcal{D}}$ is preserved.

4.3 Condition for late time acceleration

The expression for the volume $V_{\mathcal{D}}(t)$ in (36) allows us to determine the late time behaviour of the effective scale factor $a_{\mathcal{D}}(t) \equiv (V_{\mathcal{D}}(t)/V_{\mathcal{D}}(t_{in}))^{1/3}$. Using a binomial expansion for $t \gg t_{in}$ in (36), we get

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = \frac{\ddot{V}_{\mathcal{D}}}{V_{\mathcal{D}}} - \frac{2}{3}\left(\frac{\dot{V}_{\mathcal{D}}}{V_{\mathcal{D}}}\right)^2 = \frac{2\lambda_r^2}{\mathcal{I}_E t^4}\left(\mathcal{I}_{Er^2} - \frac{1}{3\mathcal{I}_E}\left(\mathcal{I}_{Er}\right)^2\right) + \mathcal{O}\left(3\right) \tag{38}$$

where $\mathcal{O}\left(3\right)$ represents terms involving $\lambda_r^m \lambda_t^n$, i.e. containing $(1/t^{m+n})$ with $m + n \geq 3$.

We see that the generic late time (i.e. $t \to \infty$) behaviour of the unbound models under consideration is $\ddot{a}_{\mathcal{D}} \to 0$, and that deviations from zero are small, being a second order effect. Whether the approach to $\ddot{a}_{\mathcal{D}} = 0$ is via an accelerating or decelerating phase, depends upon the relative magnitudes of the domain

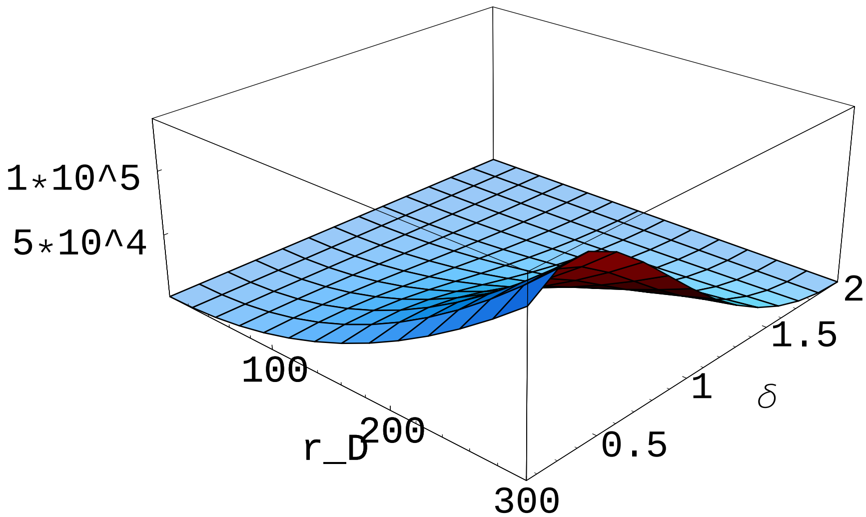


Figure 1: The function $\mathcal{P}$ defined in the text, for the power law models described by $2E(r) = r^{\delta}$. $\mathcal{P}$ is positive everywhere except along $\delta = 2$.

integrals involved. A sufficient condition for an unbound model with negligible matter to accelerate at late times, is

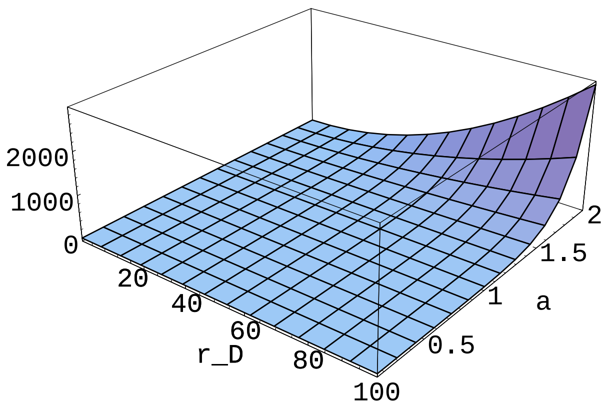
$$\mathcal{P} \equiv \mathcal{I}_{Er^2} - \frac{1}{3\mathcal{I}_E} (\mathcal{I}_{Er})^2 > 0 \,. \tag{39}$$

To proceed further we need to specify a particular model.As an explicit example of models admitting acceleration, we consider the power law models characterized by $2E(r) = r^{\delta}$, for all r, in some units. (At present we are only demonstrating the existence of such models, and shall therefore not worry about the physical scales involved.) Keeping in mind the discussion of Secs. 4.1 and 4.2, we must strictly speaking only consider the model with $\delta = 2$. The models with $\delta > 2$ cannot be considered at all, since they violate the conditions assumed in Sec. 4.2 which justified the approximation in Eqn. (27). The models with $\delta < 2$ on the other hand, contain a Ricci scalar that diverges and a matter density that vanishes at the center. Despite these pathologies, we display the results for the models with $\delta < 2$ as an existence proof of acceleration using this very simple parametrization. Although it is possible to obtain analytical expressions for the integrals in (37) in terms of the incomplete Beta function, it serves our purpose much better to numerically evaluate the integrals for various values of δ and $r_{\mathcal{D}}$, and plot the function $\mathcal{P}$ defined in (39). The results are shown in fig.1. Note that $\mathcal{P}$ vanishes along the line $\delta = 2$, but is positive *everywhere else* in the region plotted, and that the positivity of $\ddot{a}_{\mathcal{D}}/a_{\mathcal{D}}$ at late times is independent of the size of the domain $r_{\mathcal{D}}$. We have therefore obtained a continuous range of parameter values $(\delta, r_{\mathcal{D}})$ which admit late time acceleration.

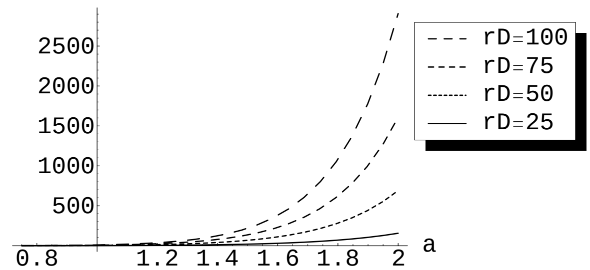
In order to demonstrate that the acceleration obtained above is not an artifact of the singular behaviour of those models, we construct another set of models which show late time acceleration, and in which the spatial Ricci scalar ${}^{(3)}\mathcal{R}$ remains finite everywhere. Consider the models characterized by the energy function

$$2E(r) = \frac{r^2}{1 + r^a} \quad ; \quad a > 0 \tag{40}$$

where we have again used arbitrary units. Since $a > 0$, the $r \to 0$ behaviour of these models is $2E \sim r^2$, which satisfies the regularity conditions of Sec. 4.1 and keeps ${}^{(3)}\mathcal{R}$ finite at the origin. Also, keeping $a < 2$ ensures the 'no shell-crossing' criterion of Sec. 4.1. The function $\mathcal{P}/\mathcal{I}_E$ for these models, which controls



(a) $\mathcal{P}/\mathcal{I}_E$



(b) $\mathcal{P}/\mathcal{I}_E$

Figure 2: The models described by $2E(r) = r^2/(1 + r^a)$. (a) The scaled function $\mathcal{P}/\mathcal{I}_E$. (b) $\mathcal{P}/\mathcal{I}_E$ plotted against a for specific values of $r_{\mathcal{D}}$.

the magnitude of the late time acceleration, is shown in fig. 2, against $r_{\mathcal{D}}$ and a. For clarity, in the second panel we have shown $\mathcal{P}/\mathcal{I}_E$ against a for specific values of $r_{\mathcal{D}}$. Again, we find that $\mathcal{P}/\mathcal{I}_E$ is positive *everywhere* in the region shown, and hence the models show late time acceleration for all allowed values of a and $r_{\mathcal{D}}$. As an example, we plot the evolution of the dimensionless quantity $q_{\mathcal{D}}$ defined by

$$q_{\mathcal{D}} \equiv -\frac{\ddot{a}_{\mathcal{D}} a_{\mathcal{D}}}{\dot{a}_{\mathcal{D}}^2} = 2 - 3\frac{\ddot{V}_{\mathcal{D}} V_{\mathcal{D}}}{\left(\dot{V}_{\mathcal{D}}\right)^2} \tag{41}$$

for various fixed values of a and $r_{\mathcal{D}}$. The results are shown in fig. 3. We have used units in which $t_{in} = 1$, and have displayed the evolution for times $t > 100\, t_{in}$. As mentioned earlier, a potentially contentious issue is that in all of the calculations above, we have actually set $\epsilon = 0$. Since the function $t_0(r)$ approaches its approximation in Eqn. (27) in a continuous fashion as $\epsilon \to 0$, we expect that models with a non-zero but small matter density will also exhibit the same qualitative late time behaviour as the ones above. To demonstrate this, we consider the leading corrections to the function $t_0(r)$ in the presence of a small but non-zero ϵ. We are still assuming the late time limit so that the ϵ dependent terms on the *right* hand side of Eqn. (33) can be neglected (more precisely, we treat both ϵ *and* g/R as small quantities). First, let us rewrite Eqn. (32) as

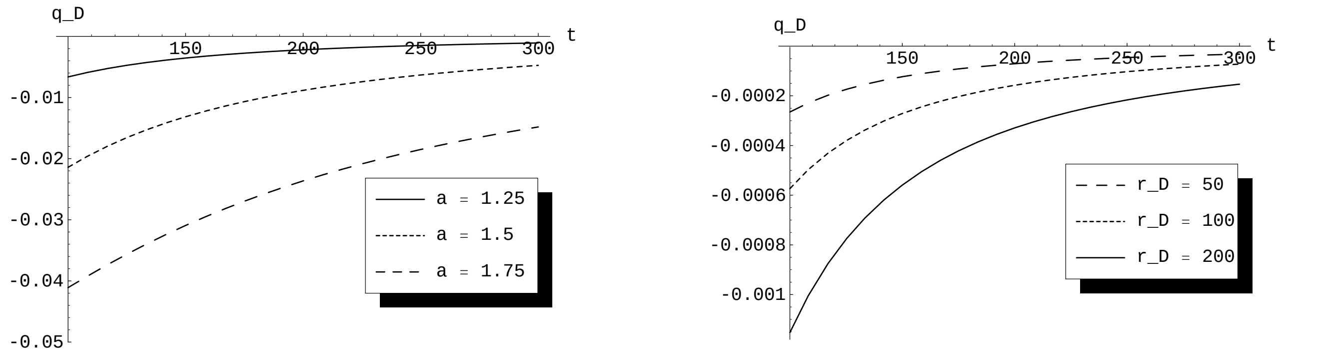
$$t_0(r) = t_{in} - \frac{h(r)}{\sqrt{2E}} \quad ; \quad h(r) \equiv r\left(\frac{Er}{GM}\right)^{1/2} F\left(\frac{Er}{GM}\right) = r\left\{1 + \frac{\epsilon g}{2r}\ln\left(\frac{\epsilon g}{r}\right) + \mathcal{O}\left(\frac{\epsilon g}{r}\right)\right\}. \tag{42}$$

It is easy to check that in the late time limit, the expression for volume $V_{\mathcal{D}}$ becomes

$$V_{\mathcal{D}} = (t - \lambda_t t_{in})^3 \mathcal{I}_E + \lambda_r (t - \lambda_t t_{in})^2 \mathcal{I}_{Eh} + \lambda_r^2 (t - \lambda_t t_{in}) \mathcal{I}_{Eh^2} + \lambda_r^3 \mathcal{I}_{h^2} \tag{43}$$

where $\mathcal{I}_E$ is the same as defined in (37), and the remaining integrals are defined analogously to those in (37),

$$\mathcal{I}_{Eh} = 4\pi \int_0^{r_{\mathcal{D}}} \frac{(2Eh)'}{\sqrt{1+2E}}\, dr \quad ; \quad \mathcal{I}_{Eh^2} = 4\pi \int_0^{r_{\mathcal{D}}} \frac{(h^2\sqrt{2E})'}{\sqrt{1+2E}}\, dr \quad ; \quad \mathcal{I}_{h^2} = 4\pi \int_0^{r_{\mathcal{D}}} \frac{h^2 h'}{\sqrt{1+2E}}\, dr\,. \tag{44}$$



(a) Domain range $r_{\mathcal{D}} = 250$

(b) Exponent $a = 1$

Figure 3: Evolution of $q_{\mathcal{D}}$ in the models with $2E(r) = r^2/(1+r^a)$, plotted for (a) three values of a with $r_{\mathcal{D}} = 250$, and (b) three values of $r_{\mathcal{D}}$ with $a = 1$.

In arriving at Eqn. (43), we have neglected terms involving $(g/R)(\epsilon \ln \epsilon)$, $\epsilon(g/R)\ln(g/R)$ and terms of order $\mathcal{O}(\epsilon g/R)$ on the right hand side of Eqn. (33). In order to proceed as before, we further assume that these leading order corrections are *smaller* than the terms of order λ_r^2 coming from the binomial expansion of $V_{\mathcal{D}}$ in Eqn. (43). This is essential in order to be able to make a statement analogous to (39), and can be ensured by choosing ϵ small enough, without setting it exactly to zero. The condition for late time acceleration in this situation becomes

$$\mathcal{P}_h \equiv \mathcal{I}_{Eh^2} - \frac{1}{3\mathcal{I}_E}\left(\mathcal{I}_{Eh}\right)^2 > 0 \,. \tag{45}$$

For $\epsilon < e^{-1}$, we have $0 < \epsilon < -\epsilon \ln \epsilon < 1$, and the leading order terms in the expansion of $h(r)$ contain $(\epsilon \ln \epsilon)$ (assuming that the r dependent coefficients are well behaved for all r). On expanding the integrals in (45) to this leading order, we find for the function $\mathcal{P}_h$,

$$\mathcal{P}_h = \mathcal{P} + (-\epsilon \ln \epsilon)\mathcal{P}^{(\epsilon \ln \epsilon)} + \mathcal{O}\left(\epsilon, (\epsilon \ln \epsilon)^2\right) \quad ; \quad \mathcal{P}^{(\epsilon \ln \epsilon)} \equiv \frac{2}{3\mathcal{I}_E}\mathcal{I}_{Er}\mathcal{I}_{Er}^{(\epsilon \ln \epsilon)} - \mathcal{I}_{Er^2}^{(\epsilon \ln \epsilon)}$$
$$\mathcal{I}_{Er}^{(\epsilon \ln \epsilon)} = 4\pi \int_0^{r_{\mathcal{D}}} \frac{G\tilde{M}'}{\sqrt{1+2E}}\, dr \quad ; \quad \mathcal{I}_{Er^2}^{(\epsilon \ln \epsilon)} = 4\pi \int_0^{r_{\mathcal{D}}} \frac{(2G\tilde{M}r/\sqrt{2E})'}{\sqrt{1+2E}}\, dr \tag{46}$$

where $\tilde{M}$ is defined by $M = \epsilon \tilde{M}$, $\mathcal{P}$ is defined in (39) and the integrals $\mathcal{I}_{Er}^{(\epsilon \ln \epsilon)}$ and $\mathcal{I}_{Er^2}^{(\epsilon \ln \epsilon)}$ give the leading order corrections to $\mathcal{I}_{Er}$ and $\mathcal{I}_{Er^2}$ respectively. The function $\mathcal{P}_h$ of (45) replaces $\mathcal{P}$ in Eqn. (38). This shows that a non-zero ϵ brings in an additional correction to $\ddot{a}_{\mathcal{D}}/a_{\mathcal{D}}$ which is of order $\lambda_r^2(\epsilon \ln \epsilon)$. We have already neglected terms of order $(g/R)(\epsilon \ln \epsilon)$, and since (g/R) is small *because* t is large, we should therefore also ignore terms of order $\lambda_r(\epsilon \ln \epsilon)$, $\lambda_t(\epsilon \ln \epsilon)$ and higher. Hence the correction given by $\mathcal{P}^{(\epsilon \ln \epsilon)}$ (provided it is finite), should be ignored. We therefore see explicitly that within the late time approximation, we can always have a non-zero but small enough ϵ which does not affect the sign of the acceleration at the leading order.

A rigorous argument to demonstrate acceleration in models with non-zero matter in general would, of course, require a complete numerical evolution of the LTB equations, and this work is in progress. Our semi-numerical analysis, however, throws open the possibility of a large class of models which may show late time acceleration after averaging.

Our results for the zero matter limit also point to the need for caution in interpreting acceleration - by suitable choices of the energy function $E(r)$ one could obtain acceleration, even though in this limit the spacetime coincides with Minkowski spacetime. A similar demonstration was earlier given by Ishibashi and Wald [5]. They showed that by suitably joining two negative curvature slices (hyperboloids) in Minkowski spacetime one can construct a spatial region which exhibits acceleration. In contrast though, the slicing we choose is physically motivated, so as to coincide with the FRW slicing when matter is included. Our own interest of course was in demonstrating, by adding matter beyond the Minkowski limit, that acceleration is possible in a single expanding LTB region.

4.4 An analytical example ($r = 0$ excluded)

In this subsection we will follow a slightly different approach and try to construct an accelerating model from purely analytic arguments. We begin with a domain in which $t_0(r) > 0$ for all r. We now use the approximate solution (27) in the expression for volume in (18) and keep the integration limits unspecified as r_1, r_2, i.e. $V_{\mathcal{D}} = 4\pi \int_{r_1}^{r_2} R'R^2/\sqrt{1+2E}dr$. At late times t, by treating $t_0(r)$ in its entirety as a small quantity compared to t, we find

$$\ddot{a}_{\mathcal{D}} \propto \frac{1}{\mathcal{I}_1 t^4}\left(\mathcal{I}_1\mathcal{I}_3 - \frac{\mathcal{I}_2^2}{3}\right) \tag{47}$$

where the constant of proportionality is positive, ϵ dependent terms have been neglected and we have defined the integrals[2]

$$\mathcal{I}_1 = \frac{1}{3}\int_{r_1}^{r_2} \frac{\left((2E)^{3/2}\right)'}{\sqrt{1+2E}}\,dr \quad ; \quad \mathcal{I}_2 = -\int_{r_1}^{r_2} \frac{\left((2E)^{3/2}t_0\right)'}{\sqrt{1+2E}}\,dr \quad ; \quad \mathcal{I}_3 = \int_{r_1}^{r_2} \frac{\left((2E)^{3/2}t_0^2\right)'}{\sqrt{1+2E}}\,dr\,. \tag{48}$$

Note that $\ddot{a}_{\mathcal{D}}$ is still a second order quantity at late times. The integral $\mathcal{I}_1$, which is essentially the same as $\mathcal{I}_E$ defined in (37), is of the form $\int x^{1/2}/\sqrt{1+x}dx$. This can be evaluated exactly and $\mathcal{I}_1$ is positive provided $E(r)$ is an increasing function of r, which we henceforth assume. Let us also assume that $t_0(r)$ is an increasing function of r, this requires $E(r)$ to increase faster than r^2. Although this is not consistent with our arguments of Sec. 4.2, notice that simultaneously requiring $t_0 > 0$ and $t_0' > 0$ places a restriction on the minimum value that r can take. For example, if $2E = (r/r_0)^m$, $m > 2$ then these conditions imply (using Eqn. (27)) that $r \geq r_1 > (r_0^m/t_{in}^2)^{1/(m-1)}$. Since the origin is necessarily excluded, the function $GM/(Er)$ will remain finite for all allowed values of r, even though $E(r)$ rises faster than r^2. So the assumptions on which the later arguments were built, are not violated.

The condition for positivity of $\ddot{a}_{\mathcal{D}}$ at late times is now $\tilde{\mathcal{P}} \equiv \mathcal{I}_1\mathcal{I}_3 - \mathcal{I}_2^2/3 > 0$. Our assumptions above ensure that the integrands of $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ are all positive. We denote

$$f \equiv (2E)^{3/2} \quad ; \quad p \equiv \frac{t_0}{t_{in}} = 1 - \frac{r}{t_{in}(2E)^{1/2}}\,. \tag{49}$$

The positivity condition can be written as

$$\left\{\int_{r_1}^{r_2} \frac{f'}{3\sqrt{1+2E}}\,dr\right\}\left\{\int_{r_1}^{r_2} \frac{t_{in}^2(fp^2)'}{\sqrt{1+2E}}\,dr\right\} > \left\{\int_{r_1}^{r_2} \frac{t_{in}(fp)'}{\sqrt{3}\sqrt{1+2E}}\,dr\right\}^2\,. \tag{50}$$

A set of sufficient conditions for this relation to hold, is

$$f' > (fp)' \quad ; \quad (fp^2)' > (fp)'\,. \tag{51}$$

Some straight-forward algebra reduces these conditions to

$$\frac{f'}{f} > \frac{p'}{1-p} \quad \text{and} \quad \frac{f'}{f} > \frac{p'}{1-p}\frac{2p-1}{p} \tag{52}$$

[2]Factors of 4π have been absorbed in the proportionality constant.

respectively. By definition, $0 < p < 1$ since we require $t_0 > 0$ for all r. This implies $(2p-1)/p < 1$ for all r, so if the first relation in (52) holds, then so does the second. The first relation, in terms of $E(r)$, reduces to $E'/E > (-1/r)$ which is necessarily true. Hence we have obtained a class of solutions which show late time acceleration of the effective scale factor $a_{\mathcal{D}}$, with the caveat that we must exclude a sphere around the origin from our domain of integration, the radius of this sphere being determined by the form of $E(r)$.

5 Discussion

We have shown that it is possible within the framework of classical General Relativity, to construct models of universes in which the average behaviour of spatial slices is that of accelerating expansion. Although the LTB models are unrealistic (since they place us at the center of the Universe), they are useful in building intuition. Especially, since LTB is an exact solution, it helps towards a deeper understanding of averaged inhomogeneous cosmological solutions of Einstein equations. In particular, our solution could be assumed to apply not necessarily to the whole Universe, but only to a local underdense region such as a void. The volume average of the late Universe is dominated by voids, and structures occupy a tiny fraction of the volume. The average over such a distribution does not lead to a FRW model. The curvature of voids can be estimated to be proportional to minus the square of their Hubble expansion rate and thus must be negative [27]. The negative curvature LTB model discussed in this paper could be useful for describing such a situation.

In both the examples which we gave, namely the power law models and the models given by Eqn. (40), the qualitative behaviour of the evolution of the effective scale factor was independent of the size of the averaging domain. Whereas the power law models were pathological in the sense that, among other things, their spatial Ricci scalars diverged at the symmetry center, the second class of models described by (40) had no such pathology.

The point to be emphasized, though, is that our analysis clearly shows the importance of curvature (expressed as a non-zero energy function $E(r)$) in causing the late time acceleration. The fact that the limit of vanishing matter density ($\epsilon \to 0$) is well defined, means that the qualitative behaviour of curvature dominated models is not expected to change by adding a finite but small amount of matter.

A few remarks comparing the results of the present paper with the concordance model in standard cosmology are in order. One could assert that observational data show that our currently accelerating universe has zero spatial curvature. On the other hand, our LTB model with zero spatial curvature (the marginally bound case) shows no acceleration. It might hence appear that our curvature dominated LTB model is of very limited interest. However, observations in the late Universe must not be matched only with a zero-curvature Universe. This assumption may be good in the early stages of evolution, but it is just a fitting ansatz to the late-time inhomogeneous Universe. What the consideration of averaged inhomogeneities in the present and other similar papers demonstrates is that the real Universe could in principle have negative curvature and yet exhibit acceleration. If this were indeed to be the case, it could eliminate the need for a dark energy.

It might also be said that observational data show that the Universe has up to thirty percent of its content in luminous and dark matter and hence our low density curvature dominated model showing acceleration does not meet this criterion. However, in reality our model is in disagreement with the concordance model, which while being one of the simplest and most favoured possibilities, need not turn out to be the final answer. The consideration of averaged inhomogeneities shows that low density, curvature dominated models can also in principle fit the observational data. This issue should hence be regarded as an open one. Furthermore, as noted above, our LTB model could by itself be considered relevant for describing a locally underdense region such as a void.

Our results highlight the intimate connection between the averaged spatial curvature, and the evolution of the kinematic back-reaction, as anticipated from the integrability condition given by Eqn. (12). It will be interesting to consider the more general dust models described by the metric (1), and enquire if acceleration can again be produced in the approximation where the averaged three-curvature dominates over the averaged matter density during some epoch of cosmic evolution.

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References

[1] Ellis, G. F. R., in *General Relativity and Gravitation* (D. Reidel Publishing Co., Dordrecht), Eds. B. Bertotti et al., (1984).

[2] Riess, A. G., et.al., *Astron. J.*, **116**, 1009, (1998)
Perlmutter, S., et.al., *Astrophys. J.*, **517**, 565, (1999).

[3] Räsänen, S., *JCAP*, **02**, 003, (2004), Notari, A., astro-ph/0503715, and references therein.

[4] Kolb, E. W., Matarrese, S. and Riotto, A., astro-ph/0506534.

[5] Ishibashi, A. and Wald, R. M., *Class. Quant. Grav.*, **23**, 235, (2006).

[6] Buchert, T., *Gen. Rel. and Grav.*, **32**, 105, (2000).

[7] Buchert, T., *Class. Quant. Grav.*, **23**, 819, (2006).

[8] Buchert, T., Larena, J. and Alimi, J-M, gr-qc/0606020.

[9] Lemaître, G. *Ann. Soc. Sci. Brux.*, A **53**, 51, (1933) (in French)
Lemaître, G. *Gen. Rel. and Grav.*, **29**, 5, (1997) (reprint).
Tolman, R. C., *Proc. Nat. Acad. Sci*, **20**, 169, (1934).
Bondi, H., *Mon. Not. R. Astron. Soc.*, **107**, 410, (1947).

[10] Celeriér, M. N., *Astron. and Astrophys.*, **353**, 63, (2000).

[11] Alnes H., Amazguioui, M. and Gron, O., astro-ph/0512006 (to appear in *Phys. Rev. D*).

[12] Vanderveld, R. A., Flanagan, E. E. and Wasserman, I., astro-ph/0602476.

[13] Sugiura, N., Nakao, K. and Harada, T., *Phys. Rev. D*, **60**, 103508, (1999).

[14] Mustapha, N., Hellaby, C. and Ellis, G. F. R., *Mon. Not. R. Astron. Soc.*, **292**, 817, (1997).

[15] Iguchi, H., Nakamura, T. and Nakao, K., *Prog. of Theo. Phys.*, **108**, 809, (2002).

[16] Nambu, Y., and Tanimoto, M., gr-qc/0507057.

[17] Moffat, J. W., astro-ph/0505326

[18] Mansouri, R., astro-ph/0512605

[19] Chuang, C. H., Gu, J. A. and Hwang, W. Y. P., astro-ph/0512651

[20] Räsänen, S., *JCAP*, **11**, 010, (2004).

[21] Apostolopoulos, P., et al., astro-ph/0603234.

[22] Buchert, T., *Gen. Rel. and Grav.*, **33**, 1381, (2001).

[23] Buchert, T., Kerscher, M. and Sicka, C., *Phys. Rev. D*, **62**, 043525, (2000).

[24] Singh, T., P. and Joshi, P., S., *Class. Quant. Grav.*, **13**, 559, (1996)

[25] Hellaby, C., *Phys. Rev. D*, **49**, 6484, (1994).

[26] Abramowitz, M. and Stegun, I., A., Eds., *Handbook of Mathematical Functions*, National Bureau of Standards, 1965.

[27] Räsänen, S., astro-ph/0607626.